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AN ALGORITHM FOR A PIECEWISE LINEAR MODEL OF TRADE AND PRODUCTI--ETC(U)
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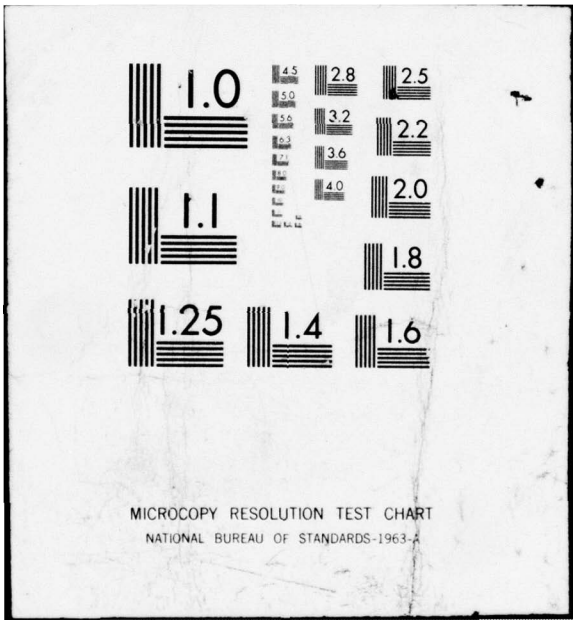
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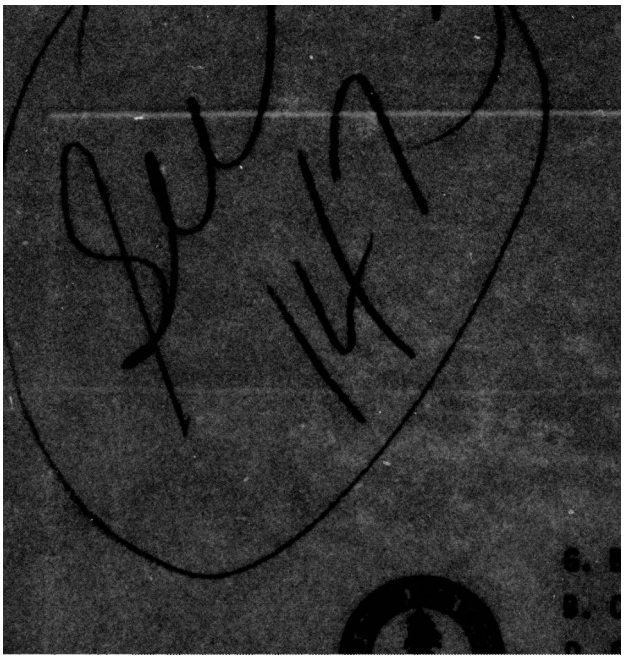

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AN ALGORITHM FOR A PIECEWISE LINEAR MODEL OF TRADE AND PRODUCTION WITH NEGATIVE PRICES AND BANKRUPTCY

August 1976

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Outline

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1. Introduction

Our interest is in computing equilibria for the Walrasian general equilibrium market where there are a large number of goods (say, two hundred) and a small number of traders (say, five). The production and consumption sets are assumed to be closed polyhedral convex sets and utilities are assumed to be piecewise linear and concave; the algorithm is designed to take as much advantage of this "convex piecewise linear structure" as our knowledge enables. More general models of trade with production and consumption sets that are merely closed and convex and with convex preference orderings in lieu of concave utilities can be approximated very closely by our "convex piecewise linear model" of trade, see, for example, Debreu [7], Kannai [16], and Mas-Colell [20].

Our approach to the problem is motivated from a number of sources, namely, the attempt of Mantel [19] to solve the "convex piecewise linear" model of trade, the fixed point methods of Scarf [24] for computing economic equilibria, a model of Mas-Colell's in [9] which shows that the equilibria cannot, in general, be computed in a finite number of steps, the existence proofs as those of Arrow and Hahn [1], Negishi [21] and Shapley [25] which are based upon the household simplex rather than the price simplex, the existence proof of Kreps [17] which uses the labeling theorem of [8] and labels based upon payments, and last but not least, the computational results of Ginsburgh and Waelbroeck [12,13,14]. The later reported surprising convergence rates by using the simplex method together with, in effect, manual adjustments of the weights on utilities to obtain an

approximate equilibrium via welfare optima. In a similar manner our algorithm uses the simplex method, but uses new fixed point methods, namely [8], to adjust these weights automatically and to thereby compute an approximate equilibrium or show that the traders are not resource related.

The model of trade we have adopted encompasses in a natural way negative or zero prices and bankruptcy. Our treatment of bankruptcy is in the sense of Arrow and Hahn [1]. Our development will utilize the notion of quasi-equilibrium or compensated equilibrium as in Debreu [5] and Arrow and Hahn [1].

The authors would like to express indebtedness to Gerard Debreu and Robert Wilson for helpful discussions. Wilson [26] has recently announced an approach for solving the "convex piecewise linear" model of trade; our approach is quite different from his especially in view of the fact that the methods were conceived over the same period amidst considerable interaction.

It is assumed that the reader is familiar with the simplex method and the duality theory of linear programming, see Dantzig [3].

2. Traders, Goods, and the Market

Consider a market with n traders and m goods; let i and j in $\nu \triangleq [1, \ldots, n]$ and g in $\mu \triangleq [1, \ldots, m]$ index the traders and goods, respectively. We note that the use of the word goods is in a very broad sense, see, for example, Debreu [4].

Each trader is endowed with a bundle of goods b_i (a $1 \times m$ vector); a sociology, technology, and productive capability A_i (an $m \times h_i$ matrix); and a utility c_i (a $1 \times h_i$ vector). Without access to the market, each trader is modeled as a linear program

$$(1) \qquad \begin{aligned} &\max_{x_i}: \ c_i x_i \\ &\text{s/t}: \ A_i x_i = b_i \qquad\qquad x_i \geq 0 \end{aligned}$$

This linear program may be infeasible reflecting, for example, that domestic agriculture may be insufficient to feed the population. By convention we shall take the gth component of b_i, b_{ig}, to be positive, zero, or negative according to whether the trader has a positive, zero, or negative inventory of the gth good.

A price π_g for good g may be positive, zero, or negative. Given access to goods of the market at prices $\pi = (\pi_1, \ldots, \pi_m)$ trader i is modeled as the linear program:

$$
\begin{aligned}
&\max_{x_i}: \ c_i x_i \\
(2) \qquad &\text{s/t}: \ A_i x_i = b_i + y_i \\
&\qquad \pi y_i \leq 0 \qquad\qquad x_i \geq 0
\end{aligned}
$$

Again the trader attempts to maximize his utility, but now he is permitted to buy and sell in the market to enhance his effort. The vector y_i is the vector of net trades of trader i with the rest of the market, and the budget constraint $\pi y_i \leq 0$ requires that he not spend more for goods than he earns in the market. The program (2) may be infeasible or even yield utility unbounded above.

For a given π and x_i we shall refer to $\pi y_i = \pi(A_i x_i - b_i)$ as the (net) payments of trader i. According to whether the payments are nonpositive ($\pi y_i \leq 0$), zero ($\pi y_i = 0$), or nonnegative ($\pi y_i > 0$) we shall say that trader i is within his budget, has balanced his budget, or is bankrupt.

An equivalent statement of (2) is

$$
\begin{aligned}
&\max_{x_i}: \ c_i x_i \\
(3) \qquad &\text{s/t}: \ \pi A_i x_i \leq \pi b_i \qquad\qquad x_i \geq 0
\end{aligned}
$$

Clearly x_i solves (3), if and only if (x_i, y_i) solves (2) where $y_i = A_i x_i - b_i$.

So far it appears that our model permits only linear utility, but we show in Section 6 that convex piecewise linear utilities are a special case. Also in Section 6 we show that production is included even though it does not appear in our model explicitly.

3. Quasi-Equilibria

Let us define our notion of a solution to the market, namely, a quasi-equilibrium (with bankruptcy) (see, for example, the compensated equilibrium of Arrow and Hahn [1]). By a partition $\alpha \cup \beta$ of the trader set ν we mean that $\alpha \cup \beta = \nu$, $\alpha \cap \beta = \emptyset$, $\alpha \neq \emptyset$, and β may or may not be empty. Let $x = (x_1, \ldots, x_n)$.

We define $(\bar{\alpha}, \bar{\beta}, \bar{x}, \bar{\pi})$ to be a quasi-equilibrium if

a) $\bar{\alpha} \cup \bar{\beta}$ partitions the trader set,

b) $\bar{x}$ is feasible, that is, $\sum_\nu A_i \bar{x}_i = \sum_\nu b_i$ and $\bar{x}_i \geq 0$,

c) each $\bar{\alpha}$ trader has nonpositive payments and subject to these payments he has maximized his utility, that is, $\bar{\pi}(A_i \bar{x}_i - b_i) \leq 0$, and $\bar{x}_i$ maximizes $c_i x_i$ subject to $\bar{\pi}(A_i x_i - b_i) \leq \bar{\pi}(A_i \bar{x}_i - b_i)$ with $x_i \geq 0$, and

d) Each β trader has nonnegative payments and has minimized his payments, that is, $\bar{\pi}(A_i \bar{x}_i - b_i) \geq 0$ and $\bar{x}_i$ minimizes $\bar{\pi}(A_i x_i - b_i)$ subject to $x_i \geq 0$.

Observe that both $(\bar{\alpha}, \bar{\beta}, \bar{x}, \bar{\pi})$ and $(\alpha', \beta', \bar{x}, \bar{\pi})$ could be quasi-equilibria with $\bar{\alpha} \neq \alpha'$; the point being that minimizing payments may have by coincidence maximized the utility subject to the budget or vice versa.

If $(\bar{\alpha}, \bar{\beta}, \bar{x}, \bar{\pi})$ is a quasi-equilibrium, but $(\nu, \emptyset, \bar{x}, \bar{\pi})$ is not, then we call $(\bar{\alpha}, \bar{\beta}, \bar{x}, \pi)$ a proper quasi-equilibrium. On the other hand if $(\nu, \emptyset, \bar{x}, \bar{\pi})$ is a quasi-equilibrium then we call $(\bar{x}, \bar{\pi})$ an equilibrium; for emphasis, note that $(\bar{x}, \bar{\pi})$ is an equilibrium, if and only if $\bar{x}$ is feasible and each trader with $\bar{x}_i$ has maximized his utility subject to his budget.

Fundamental questions are:

i) When does an equilibrium exist?

ii) How does one compute an equilibrium or show that it does not exist?

Current theory does not supply complete answers, but we shall cover the major developments known to us.

In general not even a quasi-equilibrium exists much less an equilibrium. The next three conditions guarantee the existence of a quasi-equilibrium.

I) Feasibility: We say the market is feasible, if there is an $x \geq 0$ such that $\sum_\nu A_i x_i = \sum_\nu b_i$. □

II) Nonsatiation: We say that the market possess the nonsatiation property, if each c_i contains at least one positive element. □

III) Finite Utility: We say that the market has finite utility, if for each i the program

$$\text{max:} \quad c_i x_i$$

$$\text{s/t:} \quad \sum_\nu A_j x_j = \sum_\nu b_j \qquad x \geq 0$$

has a finite objective value. □

If a quasi-equilibrium is to exist, clearly the market must be feasible; the other two conditions, though not necessary, also seem inoccuous. Each of these three conditions are easily checked.

Assumption: Throughout the remainder of the paper we assume that the market is feasibile (I), possess the nonsatiation property (II), and has finite utility (III). □

In Section 5, we supply a constructive proof, in the limiting sense, to the following theorem.

Theorem 1: A quasi-equilibrium exists. □

Observe that the price π of a quasi-equilibrium is nonzero; this follows from the nonsatiation assumption and the fact that α is nonempty. Suppose that trader i is individually feasible, that is, (1) is feasible, and that $(\bar{\alpha},\bar{\beta},\bar{x},\bar{\pi})$ is a quasi-equilibrium; then trader i is not bankrupt, that is, trader i is within his budget. Otherwise $\bar{\pi}b_i < 0$ and hence, with $\bar{\pi}A_i \geq 0$ we see that (1) is not feasible. In general one cannot conclude that an individually feasible trader is of α type.

The next lemma shows, in particular, that a quasi-equilibrium without bankruptcy is α-Pareto efficient and that an equilibrium is in the core. In a quasi-equilibrium observe that an α-trader cannot achieve the same utility at a lower payment due to the nonsatiation assumption (II); also observe that trader i can be a β-trader, if and only if $\bar{\pi}A_i \geq 0$, $\bar{\pi}A_i\bar{x}_i = 0$, and $\bar{\pi}b_i \leq 0$.

Lemma 2: If $(\bar{\alpha}, \bar{\beta}, \bar{x}, \bar{\pi})$ is a quasi-equilibrium, then there does not exist an $\alpha \subseteq \bar{\alpha}$, $\beta \subseteq \bar{\beta}$, $x_i \geq 0$ for i in $\alpha \cup \beta$, and $\delta_i \geq 1$ for i in β such that

$$\bar{\pi} \sum_{\alpha \cup \beta} (A_i \bar{x}_i - b_i) \geq 0$$

$$\sum_{\alpha} A_i x_i + \sum_{\beta} \delta_i A_i x_i = \sum_{\alpha} b_i + \sum_{\beta} \delta_i b_i$$

$$c_i \bar{x}_i \leq c_i x_i \qquad \text{for all } i \text{ in } \alpha$$

and

$$c_i \bar{x}_i < c_i x_i \qquad \text{for some } i \text{ in } \alpha$$

Proof. If so, then $\bar{\pi}(A_i x_i - b_i) \geq \bar{\pi}(A_i \bar{x}_i - b_i)$ for all i in α and with a strict relation for some i in α. Since $\delta_i \bar{\pi}(A_i x_i - b_i) \geq \bar{\pi}(A_i \bar{x}_i - b_i)$ for each β-trader we have

$$\bar{\pi}\left(\sum_{\alpha} (A_i x_i - b_i) + \sum_{\beta} \delta_i (A_i x_i - b_i)\right) > 0$$

which contradicts the feasibility of x_i for i in $\alpha \cup \beta$. □

The existence of a proper quasi-equilibrum might lead one to believe that an equilibrium does not exist. For consider, the α-traders are within their budgets, and the β-traders have foresaken their utilities and focused on balancing their budgets which they are just able to do, if at all. The following example illustrates that the model may have both a proper quasi-equilibrium and an equilibrium.

$$c_1 = (1,\ 0) \qquad c_2 = (1)$$

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

One can check that $\bar{x}_1 = (1, 1)$, $\bar{x}_2 = 1$ and $\bar{\pi} = (1, -1)$ is an equilibrium. However $\bar{\alpha} = \{1\}$, $\bar{\beta} = \{2\}$, $\bar{x}_1 = (2, 0)$, $\bar{x}_2 = 0$, and $\bar{\pi} = (1, 0)$ is a proper quasi-equilibrium; the β-trader has minimized his payments and has thereby managed to balance his budget; however, he has not maximized his utility subject to his budget.

Computation of an approximate quasi-equilibrium in a finite number of steps involves some technical difficulties that we have been unable to surmount. We are thus lead to the "indirectly resource related" notion of Arrow and Hahn [1].

(IV) Resource Related: The traders of the market are defined to be (indirectly) resource related if no subset of traders is satiated by the goods owned and produced by the remaining traders. More precisely, we say the traders are resource related if for any $\bar{x}$ with

$$\sum_{\nu} A_i \bar{x}_i = \sum_{\nu} b_i$$

and any partitions $\alpha \cup \beta$ of the traders with $\beta \neq \emptyset$, there exists numbers $\delta_i \geq 1$ for $i \in \beta$ and $x \geq 0$ such that

i) $$\sum_{\alpha} A_i x_i + \sum_{\beta} \delta_i A_i x_i = \sum_{\alpha} b_i + \sum_{\beta} \delta_i b_i ,$$

ii) $c_i \bar{x}_i \leq c_i x_i$ for all i in α,

iii) $c_i \bar{x}_i < c_i x_i$ for some i in α. □

Theorem 3: If the traders are resource related, then an equilibrium exists.

Proof: There is a quasi-equilibrium $(\bar{\alpha}, \bar{\beta}, \bar{x}, \bar{\pi})$ by Theorem 1. If $\bar{\beta} \neq \emptyset$, then according to Lemma 2, the resource related condition fails for $(\bar{\alpha}, \bar{\beta}, \bar{x})$. □

The algorithm described in Section 7 will compute (in a finite number of steps) an approximate equilibrium or show that the resource related condition (IV) fails.

4. The Aggregate Program with Weighted Objective

For our proof and algorithms we consider, in the spirit of welfare economics, the aggregated production capability of the market together with an objective function obtained by weighting the traders utilities.

Let S be the $(n-1)$-simplex $\{(\theta_1, \ldots, \theta_n) \geq 0 : \sum_\nu \theta_i = 1\}$. For $\theta \in S$ consider the linear program:

$$f(\theta) = \max_x: \sum_\nu \theta_i c_i x_i$$

$$\text{(4)} \qquad \text{s/t:} \sum_\nu A_i x_i = \sum_\nu b_i \qquad x \geq 0$$

We observe that f is a finite continuous piecewise linear convex function on S.

The dual of (4) is

$$\min_\pi: \pi(\sum_\nu b_i)$$

$$\text{(5)} \qquad \text{s/t:} \quad \pi A_i \geq \theta_i c_i \qquad i \in \nu$$

For any $\theta \in S$ we know that the optimal objective values of (4) and (5) are equal and finite.

We define three point to set maps, namely, $X : S \to R^{\sum h_i}$, $\Pi : S \to R^m$, and $P : S \to R^n$. For each $\theta \in S$, $X(\theta)$ is defined to be the set of optimal solutions to (4), $\Pi(\theta)$ is defined to be the set of optimal solutions to (5), and $P(\theta)$ is defined to be the set of payment vectors $p(x,\pi,\theta)$

$$p(x,\pi,\theta) = \begin{pmatrix} \pi(A_1x_1 - b_1) \\ \vdots \\ \pi(A_nx_n - b_n) \end{pmatrix}$$

where $x \in X(\theta)$ and $\pi \in \Pi(\theta)$. We observe that $\theta_i c_i x_i = \pi A_i x_i$ for any $i \in \nu$, $x \in X(\theta)$, and $\pi \in \Pi(\theta)$. Hence $p(x,\pi,\theta)$ is also of form

$$\begin{pmatrix} \theta_1 c_1 x_1 - \pi b_1 \\ \vdots \\ \theta_n c_n x_n - \pi b_n \end{pmatrix}$$

for $x \in X(\theta)$ and $\pi \in \Pi(\theta)$. If $p = (p_1, \ldots, p_n) \in P(\theta)$, then clearly $\sum_\nu p_i = 0$.

Lemma 4: Assume $\bar{x}$ and $\bar{\pi}$ are in $X(\bar{\theta})$ and $\Pi(\bar{\theta})$. If $p_i(\bar{x},\bar{\pi},\bar{\theta}) \leq 0$ for $\bar{\theta}_i > 0$ and $p_i(\bar{x},\bar{\pi},\bar{\theta}) \geq 0$ for $\bar{\theta}_i = 0$, then $(\bar{\alpha}, \bar{\beta}, \bar{x}, \bar{\pi})$ is a quasi-equilibrium where $\bar{\alpha} = \{i : \bar{\theta}_i > 0\}$ and $\bar{\beta} = \{i : \bar{\theta}_i = 0\}$.

Proof: Obviously $\bar{\alpha} \cup \bar{\beta}$ is a partition of ν and $\bar{x}$ is feasible. For i in $\bar{\alpha}$ we have $\bar{\pi}A_i \geq \theta_i c_i$ and $\bar{\pi}A_i\bar{x}_i = \theta_i c_i \bar{x}_i$. If $\bar{\pi}A_i x_i - \bar{\pi}b_i \leq \bar{\pi}A_i\bar{x}_i - \bar{\pi}b_i$, then $\theta_i c_i \bar{x}_i = \bar{\pi}A_i\bar{x}_i \geq \theta_i c_i x_i$ and hence trader i has optimized subject to his payment level. For i in $\bar{\beta}$ we have $\bar{\pi}A_i \geq 0$, $\bar{\pi}A_i\bar{x}_i = 0$, and $\pi b_i \leq 0$. □

In particular, if we can prove that there is a $\theta > 0$ for which $P(\theta)$ contains a zero, then we have obtained existence of an equilibrium. Note that if $q \in P(\theta)$, then the task of computing an $x \in X(\theta)$ and $\pi \in \Pi(\theta)$ with $p(x,\pi,\theta) = q$ is merely a matter of solving a linear program.

The next theorem describes the continuity properties of X, Π, and P. A point to set map, say X, is called upper-semi-continuous if θ^k tends to θ, x^k in $X(\theta^k)$, and x^k tends to x, imply x is in $X(\theta)$. A point to set map, say Π, is called lower semi-continuous, if π in $\Pi(\theta)$ and θ^k tends to θ imply there exist π^k in $\Pi(\theta^k)$ with π^k tending to π. A map which is both upper and lower semi-continuous is called continuous. We shall need the following lemma which can be proved using the fact that the objective of a linear program is a continuous function of its right hand side over the feasible range; let $D^kBz \leq d^k$ where $k = 1, 2, \ldots$ be a sequence of inequality systems in the variable z.

Lemma 5: If D^k is a diagonal matrix, D^k tends to D^∞, D^∞ has a positive diagonal, d^k tends to d^∞, and $D^kBz \leq d^k$ has a solution for $k = 1, 2, \ldots$, then $D^\infty Bz \leq d^\infty$ has a solution, and for any such solution z^∞ there is a sequence z^k tending to z^∞ with $D^kBz^k \leq d^k$. □

Lemma 6: For each θ the sets $X(\theta)$, $\Pi(\theta)$, and $P(\theta)$ are non-empty, convex, and closed. X and P are upper semi-continuous, and Π is continuous.

Proof: We shall only prove that P is upper semi-continuous; the remaining facts are much easier to establish. Assuming $p^k \in P(\theta^k)$, $p^k \to p^\infty$, and $\theta^k \to \theta^\infty$ we need to show that $p^\infty \in P(\theta^\infty)$. Select $x^k \in X(\theta^k)$ and $\pi^k \in \Pi(\theta^k)$ so that $p(x^k, \pi^k, \theta^k) = p^k$. If $x^k \to x^\infty$ and $\pi^k \to \pi^\infty$, then $x^\infty \in X(\theta)$, $\pi^\infty \in \Pi(\theta)$ and $p(x^\infty, \pi^\infty, \theta^\infty) = p^\infty$, and the result is established. The difficulty arises because x^k and or π^k may not converge.

For $k = 1, 2, \ldots, \infty$ the condition that p^k is in $P(\theta^k)$ is equivalent to the requirement that (6k) have a solution

$$
(6k)\quad
\begin{aligned}
&(ai) && \theta_i^k c_i x_i - \pi b_i = p_i^k, && i \in \nu \\
&(b) && \sum_\nu A_i x_i = \sum_\nu b_i && x \geq 0 \\
&(c) && \pi\Big(\sum_\nu b_i\Big) = f(\theta^k) && \\
&(d) && \pi A_i \geq \theta_i^k c_i, && i \in \nu .
\end{aligned}
$$

Observe that by summing each (ai) and (c) we get $\sum_\nu \theta_i c_i x_i = f(\theta^k)$ since $\sum_\nu p_i^k = 0$ for $k = 1, 2, \ldots, \infty$.

Now observe that by the assumption of finite utility we cannot have $\theta_i^k c_i x_i^k \to +\infty$, and consequently, we cannot have $\theta_i^k c_i x_i^k \to -\infty$.

Pick a subsequence of k so that each θ_i^k and $\theta_i^k c_i x_i^k$ converges. Now apply Lemma 5 to (6k) where $D^k = I$, x is set equal to x^k, and where only π is regarded as a variable.

We conclude that there is a sequence $\tilde{\pi}^k$ tending to $\tilde{\pi}^\infty$ where $(x^k, \tilde{\pi}^k)$ solves (6k) for $k = 1, 2, \ldots$. If $\theta_i^k \to 0$, then clearly $\theta_i^k c_i x_i^k \to 0$. Now apply the Lemma 5 to the system formed by the (ai_j) with $\theta_{i_j}^\infty > 0$ and (b) where $\pi = \tilde{\pi}^k$, x is the only variable and

$$D^k = \begin{pmatrix} \theta_{i_1} & & & \\ & \ddots & & \\ & & \theta_{i_\ell} & \\ & & & I \end{pmatrix}.$$

We get a sequence $\tilde{x}^k \to \tilde{x}^\infty$ solving the reduced system for $k = 1, 2, \ldots, \infty$. We now have a solution $(\tilde{x}^k, \tilde{\pi}^k)$ to (6k) tending to $(\tilde{x}^\infty, \tilde{\pi}^\infty)$ and the result is complete. □

Existential and computational difficulties are caused by the fact that the $X(\theta)$, $\Pi(\theta)$, and $P(\theta)$ may be unbounded and we move to define selections $\hat{X}(\theta)$, $\hat{\Pi}(\theta)$, and $\hat{P}(\theta)$ from them. Given two ℓ-vectors u and v we say that u is lexico greater than v, if there is an $i = 1, \ldots, \ell$ such that

$$u_j = v_j \qquad \text{for } 1 \le j \le i-1$$

and

$$u_i > v_i \ .$$

Given a set of u's the lexico maximum (resp. minimum) is the u which is greatest (resp. least) in this ordering.

Let $\hat{X}(\theta)$ be the unique solution to the (lexico) linear program

$$\text{lexico max:} \quad (\sum_{\nu} \theta_i c_i x_i, \ -x)$$

$$\text{s/t:} \quad \sum_{\nu} A_i x_i = \sum b_i \ , \qquad x \ge 0 \ .$$

Here one first maximizes $\sum_{\nu} \theta_i c_i x_i$, then minimizes x_{11}, then x_{12}, etc., and finally x_{nh_n}. Obviously $\hat{X}(\theta)$ is an element of $X(\theta)$. Furthermore, it is easily verified that $\hat{X}(\theta)$ uses linearly independent columns, that is, the columns of $(A_1, \ldots, A_n)$ corresponding to positive elements of $\hat{X}(\theta)$ are linearly independent. It follows that $\hat{X}(S) = \{\hat{X}(\theta) : \theta \in S\}$ is finite.

Select A_0 an $m \times h_0$ matrix with $h_0 \ge 0$ so that $(A_0, \ldots, A_n)$ has rank m and so that

$$A_0 x_0 + \sum_{\nu} A_i x_i = \sum_{\nu} b_i$$

$$x_0 \ge 0, \ x \ge 0$$

implies $x_0 = 0$. Such an A_0 is easily computed with elementary row operations.

Let $[\epsilon]$ be the column vector $(\epsilon, \epsilon^2, \ldots, \epsilon^m)$ and let B be any $m \times m$ submatrix formed using linearly independent columns of $(A_0, \ldots, A_n)$. For each θ define $\hat{\Pi}(\theta)$ to be the unique solution to the dual of

$$\text{max:} \quad \sum_{\nu} \theta_i c_i x_i$$

$$\text{s/t:} \quad A_0 x_0 + \sum_{\nu} A_i x_i = \sum_{\nu} b_i + B[\epsilon]$$

$$x_0 \geq 0, \quad x \geq 0$$

for all small $\epsilon > 0$ where the variables are x_0 and x. Equivalently, $\hat{\Pi}(\theta)$ is defined to be the unique solution to the (lexico) linear program

$$\text{lexico min:} \quad (\pi(\sum_{\nu} b_i), \pi B)$$

$$\text{s/t:} \quad \pi A_0 \geq 0$$

$$\pi A_i \geq \theta_i c_i \qquad i \in \nu \ .$$

It follows that $\hat{\Pi}(\theta)$ is in $\Pi(\theta)$ and that $\hat{P}(\theta)$ is in $P(\theta)$ where we define $\hat{P}(\theta)$ to be $p(\hat{X}(\theta), \hat{\Pi}(\theta), \theta)$.

Lemma 7: The function $\hat{\Pi} : S \to R^m$ is continuous and nonzero.

Proof. $\hat{\Pi}(\theta) \sum_{\nu} b_i = f(\theta)$ is a continuous function of θ, by the same reasoning, $\hat{\Pi}(\theta)B$ is a continuous function of θ, and it follows that $\hat{\Pi}$ is continuous. Since $\hat{\Pi}(\theta)\, A_i \geq \theta_i c_i$ for all i, $\hat{\Pi}(\theta) \neq 0$. □

Since $\hat{X}(S)$ and $\hat{\Pi}(S)$ are bounded, $\hat{P}(S)$ is bounded. In general $\hat{X}$ and $\hat{P}$ are not continuous.

Finally, we observe that $\hat{X}(\theta)$ and $\hat{\Pi}(\theta)$ can be computed by appropriate application of the simplex method to the (lexico) linear program

$$\text{lexico max:} \quad \left(\sum_{\nu} \theta_i c_i x_i,\ -x\right)$$

$$\text{s/t:} \quad A_0 x_0 + \sum_{\nu} A_i x_i = b$$

$$x_0 \geq 0 \qquad x \geq 0$$

where b is perturbed to $b + B[\epsilon]$ for purposes of degeneracy resolution.

5. Existences of Quasi-Equilibria

Using the function $\hat{P}$, the proper labeling theorem of [8], and Lemma 4 we prove the existence of a quasi-equilibrium.

Let $\ell : S \to R^{n-1}$ be a function, continuous or not, on the (n-1)-simplex S. If the origin of R^{n-1} is in the convex hull of $\ell(C)$ for a subset C of S we say that C is <u>ℓ-complete</u>. If every neighborhood of θ is an ℓ-complete set we call θ an <u>ℓ-complete point</u>. The proof for the following result is found in [6] or [8]; this result can be regarded as equivalent to the Brouwer fixed point theorems.

<u>Lemma 8:</u> If the vertex set of S is ℓ-complete and no (n-2)-face of S is ℓ-complete, then there is an ℓ-complete point. □

Our next step is to define a labeling $\ell : S \to \{z \in R^n : \sum_\nu z_i = 0\}$ related to our market. For θ on the boundary of S define $\ell(\theta)$ by

$$\ell_i(\theta) = \begin{cases} \theta_i & \text{if } \theta_i > 0 \\ -\frac{1}{\sigma} & \text{if } \theta_i = 0 \end{cases}$$

where σ is the number of i's for which $\theta_i = 0$. For θ in the interior of S define $\ell(\theta)$ to be $\hat{P}(\theta)$.

We see that the vertex set of S is ℓ-complete and that no (n-2)-face of S is ℓ-complete, and we may conclude that there is an ℓ-complete point.

Lemma 9: For each ℓ-complete point $\bar{\theta}$ there is a $p \in P(\bar{\theta})$ with $p_i \leq 0$ if $\bar{\theta}_i > 0$ and $p_i \geq 0$ if $\bar{\theta}_i = 0$.

Proof: From the definition of an ℓ-complete point it follows that there is a subsequence of θ^{ik} in S and $\lambda^{ik} \geq 0$ for i in ν such that $\theta^{ik} \to \bar{\theta}$ for each i in ν as k = 1, 2, ... tends to ∞ and

$$\sum_{\nu} \lambda^{ik} \ell(\theta^{ik}) = 0$$

$$\sum_{\nu} \lambda^{ik} = 1$$

for k = 1, 2, ...

By rearranging terms and selecting subsequences if necessary, we may assume that $\theta^{ik} > 0$ for i in ζ and $\theta^{ik} \not> 0$ for i in η where $\zeta \cup \eta$ partitions ν. Further we may assume that $\lambda^{ik} \to \lambda^{i}$ for i in ν, $\ell(\theta^{ik}) \to p^{i}$ for i in ζ, and $\ell(\theta^{ik}) \to q^{i}$ for i in η as k = 1, 2, ... tends to ∞ where

$$p^{i} \in P(\bar{\theta})$$

and

$$q^{i}_{j} \leq 0 \qquad \text{if } \bar{\theta}_j = 0 ,$$

$$q^{i}_{j} > 0 \qquad \text{if } \bar{\theta}_j > 0 .$$

We have

$$\sum_{\zeta} \lambda^i p^i + \sum_{\eta} \lambda^i q^i = 0$$

Divide by $s = \sum_{\zeta} \lambda^i$ and use the convexity of $P(\bar{\theta})$ to get p in $P(\bar{\theta})$ with

$$p_j \geq 0 \qquad \text{if } \bar{\theta}_j = 0 \,,$$

$$p_j \leq 0 \qquad \text{if } \bar{\theta}_j > 0 \,. \qquad \square$$

Putting together Lemmas 4, 8, and 9 we have a proof of Theorem 1, namely, a quasi-equilibrium exists. Lemma 4 also yields:

Lemma 10: If there is an ℓ-complete $\bar{\theta}$ with $\bar{\theta} > 0$, then there is an equilibrium. $\square$

6. Piecewise Linear Utility and Production

Assuming production and consumption sets are polyhedral convex sets and that preferences are reflected in a concave piecewise linear utility function we shall show that the traditional model of competitive equilibrium as described, for example, by Debreu [4], is a special case of our model as stated in Sections 2 and 3. We shall incorporate production using a construction attributed to Rader [22].

Let us suppose that there are m_0 goods in the (traditional) market. Let the production set Q_0 be a set in R^{m_0} of form

$$\{E_0 z_0 : D_0 z_0 = d_0,\ z_0 \geq 0\} \quad .$$

Note that any closed polyhedral convex set can be represented in this manner. It is assumed that disposal is free

$$q_0 \leq 0 \quad \text{implies} \quad q_0 \ \text{is in} \quad Q_0$$

that production in the large is not reversible

$$\{q_0 \in Q_0 : q_0 \geq 0\} \quad \text{is bounded.}$$

Given a set of prices π_0 for the goods, the profit $\pi_0 q_0$ is to be maximized over q_0 in Q_0.

Let us suppose that there are n consumers. We assume that the consumption set Q_i of consumer i is a set in R^{m_0}, of form

$$\{E_i z_i : D_i z_i = d_i,\ z_i \geq 0\} \quad .$$

Each consumer is assumed to have an endowment of goods w_i and a concave piecewise linear utility function U_i over the consumption set defined by

$$U_i(q) = \min_{j \in \mu_i} c_{ij} q$$

where μ_i is a finite set. Observe that any concave piecewise linear utility function can be represented in this form. Further we suppose that each consumer owns a fraction $\gamma_i \geq 0$ of the production facility where $\sum_v \gamma_i = 1$.

We assume that Q_i has a lower bound

$$q_i \text{ in } Q_i \text{ implies } q_i \geq r$$

that there is no satiation consumption

$$q_i \text{ in } Q_i \text{ implies there is a}$$

$$\bar{q}_i \text{ in } Q_i \text{ with } U(\bar{q}_i) > U(q_i)$$

and that there is a q_i in Q_i such that $q_i < w_i$.

Given this statement of elements of the model we now recast it into the form of Sections 2 and 3.

Given prices π_0 consumer i's problem is to

$$\text{max:} \quad U_i(q_i)$$

$$\text{s/t:} \quad q_i \in Q_i$$

$$\pi q_i \leq \pi_0 w_i + \gamma_i \pi_0 \bar{q}_0$$

where $\gamma_i \pi_0 \bar{q}_0$ is consumer i's income from the production and where $\bar{q}_0$ maximizes $\pi_0 q_0$ subject to $q_0 \in Q_0$. This problem of consumer i can be restated as

$$\text{max:} \quad v_i$$

$$\text{s/t:} \quad v_i \leq c_{ij} q_i , \qquad j \in \mu_i$$

$$E_i z_i = q_i$$

$$D_i z_i = d_i , \qquad z_i \geq 0$$

$$E_0 z_0 = q_0$$

$$D_0 z_0 = d_0 , \qquad z_0 \geq 0$$

$$\pi_0 q_i \leq \pi_0 w_i + \gamma_i \pi_0 q_0$$

where the variables are v_i, q_i, z_i, z_0, q_i, and q_0. Observe that if the program has a solution (in view of the nonsatiation assumption) or if the consumer minimizes his payments, then the production facility is operated optimally.

We can restate consumer i's problem again as

$$\max: \quad v_i^+ - v_i^-$$

$$\text{s/t:} \quad v_i^+ - v_i^- + s_j = c_{ij} E_i z_i , \qquad j \in \mu_i$$

$$v_i^+ \geq 0, \; v_i^- \geq 0, \; s_j \geq 0$$

$$E_i z_i - \gamma_i E_0 z_0 - w_i = y_i$$

$$D_i z_i = d_i , \qquad z_i \geq 0$$

$$D_0 z_0 = d_0 , \qquad z_0 \geq 0$$

$$\pi_0 y_i \leq 0$$

where the variables are v_i^+, v_i^-, s_j, z_i, z_0, and y_i. Aggregating all these variables except y_i to form the variable x_i and defining A_i^1, A_i^2, b_i^1, b_i^2, and c_i appropriately, we can, again, restate the consumers problem as

$$\max: \quad c_i x_i$$

$$\text{s/t:} \quad A_i^1 x_i = b_i^1 + y_i$$

$$A_i^2 x_i = b_i^2 \qquad x_i \geq 0$$

$$\pi_0 y_i \leq 0$$

Now assuming A_i^2 has m_i rows we define A_i and b_i to be the matrix and vector

$$\begin{bmatrix} A_i^1 \\ 0 \\ \vdots \\ 0 \\ A_i^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{matrix} \\ \\ \\ \\ \longleftarrow \text{Rows } \left(\sum_0^{i-1} m_i\right)+1 \text{ to } \sum_0^{i} m_i \longrightarrow \\ \\ \\ \\ \end{matrix} \quad \begin{bmatrix} b_i^1 \\ 0 \\ \vdots \\ 0 \\ b_i^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\longleftarrow \text{Rows } 1 \text{ to } m_0 \longrightarrow$$

Hence, in the model of Sections 2 and 3 there are $m = \sum_0^n m_i$ goods, but only m_0 of them are traded at a quasi-equilibrium in view of the feasibility requirement.

Since there is $q_i \leq w_i$ and disposal is free, the market is feasible (I). Clearly c_i has a positive component and insatiation (II) follows. Since each consumption set is bounded below and the production is irreversible we obtain finite utility (III). Thus the model has a quasi-equilibrium.

At a quasi-equilibrium obviously the prices π_0 are nonnegative and nonzero. Since each trader has a $q_i < w_i$, the trader can easily implement a surplus budget. Thus we have, in fact, an equilibrium. By making assumptions as, for example, in Bergstrom [2] or Hart and Kuhn [15], equilibria would not necessarily yield nonnegative prices.

7. The Algorithm and ϵ-Equilibria

Assuming feasibility (I), insatiation (II), and finite utility (III) we have shown existence of a quasi-equilibrium, and our attention is now turned to computing. In general one cannot compute in a finite number of steps (additions, multiplications, and branches on signs) a quasi-equilibrium, for there are markets with rational data (A, b, c) where all quasi-equilibria have irrational components, see [9] and Section 8. However, if there are only two traders $(n = 2)$, then a quasi-equilibrium can be computed in a finite number of steps, see Gale [10]. The next logical attempt would be to compute in a finite number of steps an approximate quasi-equilibrium. However we have not been able to devise an algorithm and a satisfactory notion of an approximate quasi-equilibrium where any degree of efficiency could be anticipated. We proceed to describe an approximate equilibrium and an algorithm which, in a finite number of steps, computes an approximate equilibrium or computes (α, β, x) showing that the traders are not resource related.

Our notion of an approximate equilibrium is that of an ϵ-equilibrium. Towards normalizing the prices we let $|\pi| = \sum_1^m |\pi_i|$. Observe that if $(\bar{x}, \bar{\pi})$ is an ϵ-equilibrium, then so is $(\bar{x}, \omega\bar{\pi})$ for any $\omega > 0$ and, in particular, for $\omega = |\bar{\pi}|^{-1}$. We call $(\bar{x}, \bar{\pi})$ an ϵ-equilibrium if

a) $\bar{x}$ is feasible, that is $\sum_\nu A_i\bar{x}_i = \sum_\nu b_i$.

b) Each trader is within ϵ of balancing his budget, that is, $\bar{\pi}A_i\bar{x}_i \leq \bar{\pi}b_i + |\bar{\pi}|\epsilon$.

c) Each trader is within ϵ of optimizing subject to his budget, that is, $c_i\bar{x}_i + \epsilon$ exceeds c_ix_i for all $x_i \geq 0$ with $\bar{\pi}A_ix_i \leq \bar{\pi}b_i$.

Our algorithm consists of three parts, a scheme for generating ℓ-complete sets, a termination rule which detects an ϵ-equilibrium, and a termination rule which detects that the traders are not resource related. The scheme for generating the ℓ-complete sets is prescribed in [6] or [8] and we shall only sketch it here. Using the labeling of Section 5, the scheme generates a sequence of ℓ-complete sets $\Theta^1, \Theta^2, \ldots$ where $\Theta^k = \{\theta^{1k}, \ldots, \theta^{nk}\} \subset S$, the diameter of the Θ^k's tend to zero as k tends to infinity, and $\Theta^{k+1} \sim \Theta^k$ contains one element for all k.

Given Θ^k at iteration k a θ^k is computed, a label $\ell(\theta^k)$ is computed, a pivot is made as in the simplex method, and Θ^{k+1} is thus determined. Thus if the θ^k is interior to S, the aggregate linear program (4) must be solved to compute $\ell(\theta^k)$. The presumption is that after the Θ^k's get small the solution of the aggregate program for θ^k is easily obtained by using the simplex method to post optimize from θ^{k-1}. Also we note that the aggregate linear program will most likely have much special structure thereby lending it to the vast store of large scale linear programming techniques, see Geoffrion [11] and Lasdon [18].

If Θ^k is interior to S, then letting x^{jk} equal $\hat{X}(\theta^{jk})$ and π^{jk} equal $\hat{\Pi}(\theta^{jk})$ we have

$$(8) \qquad \sum_{j \in \nu} [\pi^{jk} A_i x_i^{jk} - \pi^{jk} b_i] \lambda^{jk} = 0 ,$$

where $\lambda^{jk} \geq 0$ and $\sum_\nu \lambda^{jk} = 1$. Define $\pi^k = \sum_\nu \pi^{jk} \lambda^{jk}$, $x^k = \sum_\nu x^{jk} \lambda^{jk}$, and $\theta^k = \sum_\nu \theta^{jk} \lambda^{jk}$.

We shall introduce two termination rules. The first detects an ϵ-equilibrium and the second detects that the traders are not resource related.

Termination Rule 1: For a given $\epsilon > 0$ if Θ^k is interior to S and both

$$\frac{(\theta_i^{jk} - \theta_i^k) c_i x_i^{jk}}{\theta_i^k} \quad \text{and} \quad \frac{(\pi^{jk} - \pi^k) A_i x_i^{jk}}{|\pi^k|}$$

have an absolute value of ϵ or less for all i and j in ν, then terminate the algorithm. □

Lemma 11: If the algorithm terminates with Rule 1 at step k, then $(\bar{x}, \bar{\pi}) = (x^k, \pi^k)$ is an ϵ-equilibrium.

Proof: Since $\sum_\nu A_i x_i^{jk} = \sum_\nu b_i$ we have $\sum_\nu A_i \bar{x}_i = \sum_\nu b_i$ and we see that $\bar{x}$ is feasible. Next it must be shown that each trader is within ϵ of meeting his budget and within ϵ of optimizing.

From (8) we have

$$\sum_j [(\bar{\pi} + (\pi^{jk} - \bar{\pi})) A_i x_i^{jk} - \pi^{jk} b_i] \lambda^{jk} = 0$$

or

$$\bar{\pi} A_i \bar{x}_i = \bar{\pi} b_i + \sum_j (\pi^{jk} - \bar{\pi}) A_i x_i^{jk} \lambda^{jk}$$

or finally

$$\bar{\pi} A_i \bar{x}_i \leq \bar{\pi} b_i + |\bar{\pi}| \epsilon$$

that is, each trader is within ϵ of meeting his budget.

From (8) and $\theta_i^{jk} c_i x_i^{jk} = \pi^{jk} A_i x_i^{jk}$ and letting $\bar{\theta} = \theta^k$ we have

$$\sum_j [(\bar{\theta}_i + (\theta_i^{jk} - \bar{\theta}_i) c_i x_i^{jk} - \pi^{jk} b_i] \lambda^{jk} = 0$$

or

$$\bar{\theta}_i c_i \bar{x}_i = \bar{\pi} b_i + \sum_j (\theta_i^{jk} - \bar{\theta}_i) c_i x_i^{jk} \lambda^{jk}$$

or finally

$$c_i \bar{x}_i \geq \frac{\bar{\pi} b_i}{\bar{\theta}_i} - \epsilon \quad .$$

We have $\pi^{jk} A_i \geq \theta_i^{jk} c_i$ and hence $\bar{\pi} A_i \geq \bar{\theta}_i c_i$. Thus at prices $\bar{\pi}$ for a trader to meet his budget we have $\bar{\pi} A_i x_i \leq \bar{\pi} b_i$, hence $\bar{\theta}_i c_i x_i \leq \bar{\pi} b_i$ or

$$c_i x_i \leq \frac{\bar{\pi} b_i}{\bar{\theta}_i} \quad .$$

Therefore, given his budget, trader i is within ϵ of optimizing. □

We regard $\bar{\theta}$ as a cluster point of the Θ^k if every neighborhood of $\bar{\theta}$ contains infinitely many of the Θ^k. Of course, the sequence Θ^k has at least one cluster point.

Lemma 12: Let ϵ be given and assume that the sequence of ℓ-complete sets Θ^k has at least one cluster point interior to S. Then the algorithm will terminate after a finite number of steps.

Proof: A k occurs for which Rule 1 will terminate the algorithm, (if Rule 2 has not already) since the x^{jk}'s come from a finite set $\hat{X}(S)$, $\hat{\Pi}$ is continuous and nonzero, the diameter of Θ^k tends to zero, and Θ^k has a cluster point interior to S. □

For the second termination rule we say that Θ^k is within ϵ of the β-face if $\theta_i^{jk} \leq \epsilon$ for all i in β and j in ν. Let K be an infinite subsequence. For θ in S let us define $\gamma(\theta)$ to be the set of (i,j) such that

$$\hat{\Pi}(\theta)\ A_{ij} = \theta_i\ c_{ij} \quad .$$

Termination Rule 2: Let δ and the subsequence K be given. If k is in K, solve the following linear program for all $(\beta, \gamma(\bar{\theta}))$ such that Θ^k is within δ of the β-face and $\bar{\theta}$ is in Θ^k.

max: z

s/t:

$$\pi A_{ij} = \theta_i c_{ij} \qquad (i,j) \in \gamma(\bar{\theta})$$

$$\pi A_{ij} \geq \theta_i c_{ij} \qquad (i,j) \notin \gamma(\bar{\theta})$$

$$\theta_i = 0 \qquad i \in \beta$$

$$\pi b_i \leq 0 \qquad i \in \beta$$

$$\theta_i \geq z \qquad i \notin \beta$$

The variables are z, θ, and π. If the optimal z is positive, terminate the algorithm. □

Lemma 13: If the algorithm terminates with Rule 2, the traders are not resource related.

Proof: If the system has a solution then we have a θ in the boundary of S and π in $\Pi(\theta)$ where $\theta_i = 0$ implies $\pi b_i \leq 0$. Hence the following lemma applies. □

Lemma 14: For fixed θ in the boundary of S and π in $\Pi(\theta)$ suppose that $\theta_i = 0$ implies $\pi b_i \leq 0$. Then the traders are not resource related.

Proof: Let $\alpha = \{i : \theta_i > 0\}$, $\beta = \{i : \theta_i = 0\}$, and $\bar{x} \in X(\theta)$. If $\delta_i \geq 1$ for i in β and $x \geq 0$ satisfies

$$\sum_{\alpha} A_i x_i + \sum_{\beta} \delta_i A_i x_i = \sum_{\alpha} b_i + \sum_{\beta} \delta_i b_i$$

we have

$$\sum_{\alpha} \theta_i c_i x_i \leq \sum_{\alpha} \pi A_i x_i$$

$$= \sum_{\alpha} \pi b_i + \sum_{\beta} \delta_i \pi b_i - \sum_{\beta} \delta_i \pi A_i x_i$$

$$\leq \sum_{\alpha} \pi b_i + \sum_{\beta} \pi b_i$$

$$= \sum_{\alpha} \pi A_i \bar{x}_i = \sum_{\alpha} \theta_i c_i \bar{x}_i \quad . \qquad \square$$

Theorem 15: Let $\epsilon > 0$, $\delta > 0$, and K be given. Then the algorithms terminates after a finite number of steps with an ϵ-equilibrium or a (α, β, x) showing the traders are not resource related.

Proof: In view of Rule 1 we can assume that Θ^k does not have any cluster points interior to S. Clearly $\gamma(S)$ is a finite set and $\Theta(\eta) = \{\theta : \gamma(\theta) = \eta\}$ is a closed polyhedral convex set for any η in $\gamma(S)$. Hence the $\Theta(\eta)$'s form a finite cover of S. Thus if Rule 2 is applied for some η in $\gamma(S)$ infinitely often, then $\Theta(\eta)$ contains a cluster point $\bar{\theta}$ of the Θ^k sequence. Suppose $\bar{\theta}$ lies in the β-face, then, in fact, Rule 2 would have terminated the algorithm for the program (β, η) the first time η was encountered, since $\bar{\theta}$ yields a quasi-equilibrium. $\square$

Obviously, if K is taken as the set of integers and δ is large or if ν is large, implementations of Rule 2 is prohibitively burdensome. The idea is, however, to determine what β-face contains in its interior a cluster point of the Θ^k and then to run the test ever-so-often for different $\gamma(\bar{\theta})$ as the Θ^k get close to the β-face.

Note that if the payments vector $p(x, \pi, \theta)$ for each θ in Θ^k is based upon the same $\bar{x}$, then $(\bar{x}, \bar{\pi})$ is an equilibrium where $\bar{\pi} = \sum \lambda^{jk} \pi^{jk}$. Also, since the sequence $X(\Theta^k \sim \Theta^{k-1})$ for $k = 2, 3, \ldots$ contains only finitely many terms, at some iteration one might be able to guess those terms which are in $X(\Theta^k)$ infinitely often. In this case one can implement a special "tail" routine in which the aggregate linear program and its dual is not resolved.

8. An Example

A preliminary version of the algorithm of Section 7 has been programmed by D. Solow and used to compute an ε-equilibrium of Mas-Colell's pure trade model of [9]. We first describe the model and then the computational results.

The trade model has three players and two goods. Each trader has one unit of each good and the utility functions are, respectively,

$$\min(x,2y)$$

$$\min(2x,y)$$

$$\min(4x,5y)$$

where x and y are the quantities of the two goods consumed. Note that equilibrium prices of this model are irrational. Recasting the problem into our format we have $c_1 = c_2 = c_3 = (1\ 0\ 0\ 0\ 0)$, $b_1 = b_2 = b_3 = (1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)$

$$A_1 = \begin{pmatrix} A_1^1 \\ A_1^2 \\ 0 \\ 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} A_1^1 \\ 0 \\ A_2^2 \\ 0 \end{pmatrix} \qquad A_3 = \begin{pmatrix} A_1^1 \\ 0 \\ 0 \\ A_3^2 \end{pmatrix}$$

$$A_1^1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad A_2^2 = \begin{pmatrix} 1 & -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$A_1^2 = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 1 \end{pmatrix} \qquad A_3^2 = \begin{pmatrix} 1 & -4 & 0 & 1 & 0 \\ 1 & 0 & -5 & 0 & 1 \end{pmatrix}$$

and each A_i is 8×5.

The computational results are summarized in the following table where k is the iteration number, ε is that of the ε-equilibrium, and s-indicates the cumulative number of simplex pivots required to solve the aggregate linear program (including Phase I for initialization).

k	ε	s
3	-	22
7	-	23
11	10.286	27
14	3.7894	30
17	1.3924	32
22	.69444	33
26	.33640	34
30	.16778	35
34	.08327	37
38	.04146	40
41	.02072	41
45	.00280	43
49	.00632	44
53	.00258	46
57	.00129	48
63	.00065	49
67	.00032	53
70	.00016	55
73	.00008	56

Hence, for example, a (.00008)-equilibrium was computed with 56 simplex pivots and 73 iterations.

The overall run time was about four-tenths of a second on an IBM 370/168. The computation behaved essentially as expected and neither confirms or denies that the algorithm is a practical device for models in the target area of five traders and two hundred goods. Applications to more serious problems will be reported upon at a later date.

Observe that the convergence rate is linear in the tail wherein the error is halfed every 4 or so iterations. If the vectors which appear in $\cup X(\Theta^k)$ infinitely often can be identified, then quadratic convergence can be obtained by using Newton's method or, for example, Saigal [23].

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An algorithm for a piecewise linear model of trade and production with negative prices and bankruptcy is described. The algorithm uses complementary pivoting to adjust the utility weights as in welfare economics and generates an approximate equilibrium or demonstrates that the traders are not resource related.

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